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Queueing Modelling of Machining System with Balking, Reneging, Additional Repairmen and Two Modes of Failure

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ABSTRACT. To improve the system reliability, the provision of spare part support is recommended in machining system. In this paper, we study M/M/R machine repair problem (MRP) with balking, reneging and additional repairmen. The units are assumed to fail in two modes with equal probability for each mode. There is provision of S standby units in the system to improve the system reliability/ availability. When a unit breaks down, it is replaced by a spare unit if available. The failed unit is repaired by a repair facility consisting of R permanent and r additional repairmen. If the number of failed units is more than the number of repairmen then the failed unit will not get repair immediately. In such circumchance, it may balk or renege. The machine repair problem is formulated as birth- death process and the steady state queue size distribution is obtained in product form. Some performance measures such as expected number of failed units, expected number of spare units and machine availability etc. are obtained. The expected total cost per unit time has also been facilitated.

KEYWORDS: Queue, Machine Interference, Balking, Reneging, Spares, Two Modes of Failure, Additional Repairmen.

1. Introduction

The machine repair modeling has become a popular approach to study various manufacturing and production systems. Machine repair problems have also been used to analyze multiprogramming computer system, communication system and time-sharing systems. The provision of spare part support is recommended in order to improve the grade of service of the system. Sometimes, due to restriction of volume/ cost, it is not feasible to provide the sufficient number of spares, which are needed for smooth running of the system. In such cases, the provision of additional repairmen may be advisable to get desired reliability / availability. Machine repair problem with spares and additional repairmen is an extension of machine interference problem. We develop a machine repair model wherein units have two failure modes and repaired in FIFO order with spares. The system consists of M operating units and S spare units. When a unit fails, it is replaced by a spare unit and is repaired by a repairman in FIFO order. The repair facility has R permanent and r additional repairmen.

The machine repair problem has been studied by many researchers. Jaiswal and Thiruvengadom [8] studied machine interference problem with two types of failures. King [10] developed the optimum size of work force engaged in the servicing of automatic machines by using the method of discrete transform. Reynolds [11] derived the M/M/m/n queue for the shortest distance priority machine interference problem. He obtained the largest number of operating machines under the steady state condition. Goel et al. [4] provided the cost analysis of a two units cold standby system with two types of operation and repair. Wang and Wu [14] considered the cost analysis of the M/M/R machine repair problem with spares and two modes of failures. Jain [6] developed the M/M/R machine repair problem with spares and additional repairmen by using queue size distribution of failed machines. Gupta [5] suggested N -policy queueing system with finite source and warm spares.

It is common that some times the queue is so long that a customer may be discouraged after joining the queue and leave the system without being served. This queueing situation is said to be reneging. If the customer decided not to join the queue, this type of queueing situation is said to be balking. In many realistic manufacturing / production situations, due to long backlog of failed units, the units may be discouraged i. e. either balked or reneged. Blackburn [3] gave optimal control of a single server queue with balking and reneging. Biswas and Sunaga [2] developed the diffusion approximation method for multi- server queueing system with balking. Abou-El-Ata and Shawky [1] derived the analytical solution of the single server Markovian over-flow queue with balking, reneging and an additional server for longer queues. Jain and Prem Lata [7] suggested M/M/R machine repair problem with reneging and spares. Shawky [12] studied the steady state solution of the single server machine repair problem with balking, reneging and an additional server for a longer queue. Ke and Wang [9] developed the cost analysis of the M/M/R machine repair problem with balking, reneging and server breakdowns. Shawky [13] considered M/M/C/K/N machine interference model with balking, reneging and spares.

We develop M/M/R machine repair problem with balking, reneging, spares, additional repairmen and two modes of failure by using birth- death process. The life times and the repair times of the units are assumed to be exponentially distributed. The queue size distribution is obtained under steady state, which is employed to derive various performance measures. Finally, we determine the expression for total expected cost per unit time for the system by using the queue size distribution.

2. Model Description and Analysis

Consider M/M/R machine repair problem wherein N = M (operating) + S (spare) units are under the care of R permanent repairmen and r additional repairmen. Each operating unit has two independent failure modes. The failure rates of operating units in modes 1 and 2 are λ_1 and λ_2 respectively. The repair rates of permanent (additional) repairmen are $\mu_1(\mu_1')$ and $\mu_2(\mu_2')$ for modes 1 and 2 respectively. When an operating unit fails, it is sent for repairing to repair facility and spare unit is used to replace the operating unit if available. The failed units are repaired by repairmen in FIFO discipline. A repairman can repair only one failed unit at a time. If all permanent repairmen are busy and a unit fails, additional repairmen are available. When the repairing of a failed unit is completed, it joins standby group and it is as good as new one. When all repairmen are busy, the failed units may balk with rates β_1 $(0 \le \beta_1 < 1)$ and β_2 $(0 \le \beta_2 < 1)$ for the units fail in mode 1 and 2 respectively. If all the repairmen are busy, the failed units may renege in exponential fashion with parameter ν_1 and ν_2 for units failed in modes 1 and 2 respectively. The steady state probability that there are i and j failed units in the system of modes 1 and 2 respectively, is represented by P(i, j). The steady state equations governing the model are constructed for two cases as follows:

Case I : R $\le$ S

In this case the steady state difference equations are

$$-[M\lambda_1 + M\lambda_2]P(0, 0) + \mu_1 P(1, 0) + \mu_2 P(0, 1) = 0 \quad (1)$$

$$-[M\lambda_1 + M\lambda_2 + i\mu_1]P(i, 0) + M\lambda_1 P(i-1, 0) + (i+1)\mu_1 P(i+1, 0) + \mu_2 P(i, 1) = 0, \quad 1 \le i < R \quad (2)$$

$$-[M\beta_1\lambda_1 + M\lambda_2 + R\mu_1]P(R, 0) + M\lambda_1 P(R-1, 0) + (R\mu_1 + \nu_1)P(R+1, 0) + \mu_2 P(R, 1) = 0, \quad (3)$$

$$-[M\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + (i-R)\nu_1]P(i, 0) + M\beta_1\lambda_1 P(i-1, 0) + [R\mu_1 + (i+1-R)\nu_1]P(i+1, 0) + \mu_2 P(i, 1) = 0, \quad R < i \le S \quad (4)$$

$$-[(N-i)\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + (i-R)\nu_1]P(i, 0) + (N-i+1)\beta_1\lambda_1 P(i-1, 0) + [R\mu_1 + (i+1-R)\nu_1]P(i+1, 0) + \mu_2 P(i, 1) = 0, \quad S < i < T \quad (5)$$

$$-[(N-T)\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + (T-R)\nu_1]P(T,0) + (N-T+1)\beta_1\lambda_1 P(T-1,0) + [R\mu_1 + \mu_1' + (T-R)\nu_1]P(T+1,0) + \mu_2 P(T,1) = 0, \quad (6)$$

$$-[(N-kT)\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + (k-1)\mu_1' + (kT-R-k+1)\nu_1]P(kT,0) + (N-kT+1)\beta_1\lambda_1 P(kT-1,0) + [R\mu_1 + k\mu_1' + (kT+1-R-k)\nu_1]P(kT+1,0) + \mu_2 P(kT,1) = 0,\ k = 1, 2, \ldots, r-1, \quad (7)$$

$$-[(N-rT)\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + (r-1)\mu_1' + (rT-R-r+1)\nu_1]P(rT,0) + (N-rT+1)\beta_1\lambda_1 P(rT-1,0) + [R\mu_1 + r\mu_1' + (rT+1-R-r)\nu_1]P(rT+1,0) + \mu_2 P(rT,1) = 0, \quad (8)$$

$$-[(N-i)\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + r\mu_1' + (i-R-r)\nu_1]P(i,0) + (N-i+1)\beta_1\lambda_1 P(i-1,0) + [R\mu_1 + r\mu_1' + (i+1-R-r)\nu_1]P(i+1,0) + \mu_2 P(i,1) = 0,\ rT<i<N \quad (9)$$

$$-[R\mu_1 + r\mu_1' + (N-R-r)\nu_1]P(N,0) + \beta_1\lambda_1 P(N-1,0) = 0, \quad (10)$$

$$-[M\lambda_1 + M\lambda_2 + j\mu_2]P(0,j) + M\lambda_2 P(0,j-1) + (j+1)\mu_2 P(0,j+1) + \mu_1 P(1,j) = 0,\ 1 \le j < R \quad (11)$$

$$-[M\lambda_1 + M\beta_2\lambda_2 + R\mu_2]P(0,R) + M\lambda_2 P(0,R-1) + (R\mu_2 + \nu_2)P(0,R+1) + \mu_1 P(1,R) = 0, \quad (12)$$

$$-[M\lambda_1 + M\beta_2\lambda_2 + R\mu_2 + (j-R)\nu_2]P(0,j) + M\beta_2\lambda_2 P(0,j-1) + [R\mu_2 + (j+1-R)\nu_2]P(0,j+1) + \mu_1 P(1,j) = 0,\ R<j \le S \quad (13)$$

$$-[M\lambda_1 + (N-j)\beta_2\lambda_2 + R\mu_2 + (j-R)\nu_2]P(0,j) + (N-j+1)\beta_2\lambda_2 P(0,j-1) + [R\mu_2 + (j+1-R)\nu_2]P(0,j+1) + \mu_1 P(1,j) = 0,\ S < j < T \quad (14)$$

$$-[M\lambda_1 + (N-T)\beta_2\lambda_2 + R\mu_2 + (T-R)\nu_2]P(0,T) + (N-T+1)\beta_2\lambda_2 P(0,T-1) + [R\mu_2 + \mu_2' + (T-R)\nu_2]P(0,T+1) + \mu_1 P(1,T) = 0, \quad (15)$$

$$-[M\lambda_1 + (N-kT)\beta_2\lambda_2 + R\mu_2 + (k-1)\mu_2' + (kT-R-k+1)\nu_2]P(0,kT) + (N-kT+1)\beta_2\lambda_2 P(0,kT-1) + [R\mu_2 + k\mu_2' + (kT+1-R-k)\nu_2]P(0,kT+1) + \mu_1 P(1,kT) = 0,\ k = 1,2,\ldots, r-1 \quad (16)$$

$$-[M\lambda_1 + (N-rT)\beta_2\lambda_2 + R\mu_2 + (r-1)\mu_2' + (rT-R-r+1)\nu_2]P(0,rT) + (N-rT+1)\beta_2\lambda_2 P(0,rT-1) + [R\mu_2 + r\mu_2' + (rT+1-R-r)\nu_2]P(0,rT+1) + \mu_1 P(1,rT) = 0, \quad (17)$$

$$-[M\lambda_1 + (N-j)\beta_2\lambda_2 + R\mu_2 + r\mu_2' + (j-R-r)\nu_2]P(0,j) + (N-j+1)\beta_2\lambda_2 P(0,j-1) + [R\mu_2 + r\mu_2' + (j+1-R-r)\nu_2]P(0,j+1) + \mu_1 P(1,j) = 0,\ rT<j<N \quad (18)$$

$$-[R\mu_2 + r\mu_2' + (N-R-r)\nu_2]P(0,N) + \beta_2\lambda_2 P(0,N-1) = 0, \quad (19)$$

$$-[M\lambda_1 + M\lambda_2 + i\mu_1 + j\mu_2]P(i,j) + M\lambda_1 P(i-1,j) + M\lambda_2 P(i,j-1) + (i+1)\mu_1 P(i+1,j) + (j+1)\mu_2 P(i,j+1) = 0,\ i,j \ne 0,\ 1 \le i+j \le R \quad (20)$$

$$-[M\beta_1\lambda_1 + M\beta_2\lambda_2 + R\mu_1 + (i-R)\nu_1 + R\mu_2 + (j-R)\nu_2]P(i,j) +$$

$$M\beta_1\lambda_1 P(i-1, j)+M\beta_2\lambda_2 P(i, j-1) +[R\mu_1+(i+1-R)\nu_1]P(i+1, j)+ [R\mu_2+ (j+1-R)\nu_2]P(i, j+1) = 0, \; i, j \neq 0, \; R< i+j \leq S \quad (21)$$

$$-[(N-i-j)(\beta_1\lambda_1+\beta_2\lambda_2) +R\mu_1+(i-R)\nu_1+R\mu_2+ (j-R)\nu_2]P(i, j) + [R\mu_1+(i+1-R)\nu_1]P(i+1, j) +[R\mu_2+ (j+1-R)\nu_2]P(i, j+1) + (N-i-j+1)[\beta_1\lambda_1 P(i-1, j) +\beta_2\lambda_2 P(i, j-1)] = 0, \; i, j \neq T \quad S < i+j \leq T \quad (22)$$

$$-[(N-i-j)(\beta_1\lambda_1+\beta_2\lambda_2) +R\mu_1+ (k-1)\mu_1'+(i-R-k+1)\nu_1+ R\mu_2+(k-1)\mu_2'+ (j-R-k+1)\nu_2]P(i, j) +[R\mu_1+k\mu_1'+ (i+1-R-k)\nu_1]P(i+1, j) + [R\mu_2+k\mu_2' +(j+1-R-k)\nu_2]P(i, j+1) + (N-i-j+1)[\beta_1\lambda_1 P(i-1, j) + \beta_2\lambda_2 P(i, j-1)] = 0, \; i, j \neq kT, \; i+j = kT, \; k=1,2,...,r-1 \quad (23)$$

$$-[(N-i-j)(\beta_1\lambda_1+\beta_2\lambda_2) +R\mu_1+ k\mu_1'+(i-R-k)\nu_1+R\mu_2+ k\mu_2'+(j-R-k)\nu_2]P(i, j) +[R\mu_1+(k+1)\mu_1'+(i-R-k)\nu_1]P(i+1, j) + [R\mu_2+(k+1)\mu_2' +(j-R-k)\nu_2]P(i, j+1) + (N-i-j+1)[\beta_1\lambda_1 P(i-1, j) +\beta_2\lambda_2 P(i, j-1)] = 0, \; kT< i+j \leq (k+1)T \quad (24)$$

$$-[(N-i-j)(\beta_1\lambda_1+\beta_2\lambda_2) +R\mu_1+ (r-1)\mu_1'+(i-R-r+1)\nu_1+ R\mu_2+(r-1)\mu_2'+ (j-R-r+1)\nu_2]P(i, j) +[R\mu_1+r\mu_1'+(i+1-R-r)\nu_1]P(i+1, j) +[R\mu_2+r\mu_2' + (j+1-R-r)\nu_2]P(i, j+1) + (N-i-j+1)[\beta_1\lambda_1 P(i-1, j) + \beta_2\lambda_2 P(i, j-1)] = 0, \; i, j \neq rT, \; i+j = rT \quad (25)$$

$$-[(N-i-j)(\beta_1\lambda_1+\beta_2\lambda_2) +R\mu_1+ r\mu_1'+(i-R-r)\nu_1+R\mu_2+r\mu_2'+ (j-R-r)\nu_2]P(i, j) +[R\mu_1+ r\mu_1'+(i+1-R-r)\nu_1]P(i+1, j) + [R\mu_2+r\mu_2' +(j+1-R-r)\nu_2]P(i, j+1) + (N-i-j+1)[\beta_1\lambda_1 P(i-1, j) +\beta_2\lambda_2 P(i, j-1)] = 0, \; i, j \neq N, \; rT< i+j \leq N \quad (26)$$

To obtain closed form expressions for the probability P(i, j), for equations (1)-(26), we use the product type solution

$$P(i,0)=\begin{cases} \dfrac{M^{i}\rho_1^{i}}{i!}P(0,0), & 1\le i\le R\\[2ex] \dfrac{M^{i}\rho_1^{i}\beta_1^{i-R}}{R!\prod\limits_{n=R+1}^{i}[R+(n-R)\delta_1]}P(0,0), & R<i\le S\\[2ex] \dfrac{M^{S}M!\rho_1^{i}\beta_1^{i-R}}{R!(N-i)!\prod\limits_{n=R+1}^{i}[R+(n-R)\delta_1]}P(0,0), & S<i\le T\\[2ex] \dfrac{M^{S}M!\rho_1^{i}\beta_1^{i-R}}{R!(N-i)!\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_1]\prod\limits_{l=1}^{k-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_1+(n-R-l)\delta_1]\prod\limits_{n=kT+1}^{i}[R+k\gamma_1+(n-R-k)\delta_1]}P(0,0), & kT<i\le(k+1)T,\ 1\le k<r\\[2ex] \dfrac{M^{i}M!\rho_1^{i}\beta_1^{i-R}}{R!(N-i)!\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_1]\prod\limits_{l=1}^{r-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_1+(n-R-l)\delta_1]\prod\limits_{n=rT+1}^{i}[R+r\gamma_1+(n-R-r)\delta_1]}P(0,0), & rT<i\le N \end{cases}\quad(27)$$

$$P(0,j)=\begin{cases}\dfrac{M^{j}\,\rho_2^{j}}{j\,!}P(0,0), & 1\le j\le R\\[2ex] \dfrac{M^{j}\,\rho_2^{j}\,\beta_2^{j-R}}{R\,!\prod\limits_{n=R+1}^{j}[R+(n-R)\delta_2]}P(0,0), & R<j\le S\\[2ex] \dfrac{M^{S}\,M\,!\,\rho_2^{j}\,\beta_2^{j-R}}{R\,!\,(N-j)\,!\prod\limits_{n=R+1}^{j}[R+(n-R)\delta_2]}P(0,0), & S<j\le T\\[2ex] \dfrac{M^{S}\,M\,!\,\rho_2^{j}\,\beta_2^{j-R}}{R\,!\,(N-j)\,!\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_2]\prod\limits_{l=1}^{k-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_2+(n-R-l)\delta_2]\prod\limits_{n=kT+1}^{j}[R+k\gamma_2+(n-R-k)\delta_2]}P(0,0), & \\ & kT<j\le(k+1)T,\ \ 1\le k<r\\[2ex] \dfrac{M^{S}\,M\,!\,\rho_2^{j}\,\beta_2^{j-R}}{R\,!\,(N-j)\,!\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_2]\prod\limits_{l=1}^{r-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_2+(n-R-l)\delta_2]\prod\limits_{n=rT+1}^{j}[R+r\gamma_2+(n-R-r)\delta_2]}P(0,0), & \\ & rT<j\le N\end{cases}\qquad(28)$$

$$P(i,j)=\begin{cases}
\dfrac{M^{i+j}\rho_1^i\rho_2^j}{i\,!\,j\,!}P(0,0), & 1\le i+j\le R\\[2ex]
\dfrac{M^{i+j}\rho_1^i\rho_2^j\beta_1^{i-R}\beta_2^{j-R}}{(R!)^2\prod\limits_{n=R+1}^{i}[R+(n-R)\delta_1]\prod\limits_{n=R+1}^{j}[R+(n-R)\delta_2]}P(0,0), & R<i+j\le S\\[2ex]
\dfrac{M^{S}M\,!\,\rho_1^i\rho_2^j\beta_1^{i-R}\beta_2^{j-R}}{(R!)^2\,(N-i-j)!\prod\limits_{n=R+1}^{i}[R+(n-R)\delta_1]\prod\limits_{n=R+1}^{j}[R+(n-R)\delta_2]}P(0,0), & S<i+j\le T\\[2ex]
\dfrac{M^{S}M\,!\,\rho_1^i\rho_2^j\beta_1^{i-R}\beta_2^{j-R}}{\begin{array}{l}(R!)^2(N-i-j)!\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_1]\prod\limits_{l=1}^{k-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_1+(n-R-l)\delta_1]\prod\limits_{n=kT+1}^{i}[R+k\gamma_1+(n-R-k)\delta_1]\\ \quad\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_2]\prod\limits_{l=1}^{k-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_2+(n-R-l)\delta_2]\prod\limits_{n=kT+1}^{j}[R+k\gamma_2+(n-R-k)\delta_2]\end{array}}P(0,0), & kT<i+j\le(k+1)T,\ 1\le k<r\\[2ex]
\dfrac{M^{S}M\,!\,\rho_1^i\rho_2^j\beta_1^{i-R}\beta_2^{j-R}}{\begin{array}{l}(R!)^2(N-i-j)!\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_1]\prod\limits_{l=1}^{r-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_1+(n-R-l)\delta_1]\prod\limits_{n=rT+1}^{i}[R+r\gamma_1+(n-R-r)\delta_1]\\ \quad\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_2]\prod\limits_{l=1}^{r-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_2+(n-R-l)\delta_2]\prod\limits_{n=rT+1}^{j}[R+r\gamma_2+(n-R-r)\delta_2]\end{array}}P(0,0), & rT<i\le N
\end{cases}\tag{29}$$

where $\rho_1 = \frac{\lambda_1}{\mu_1}$, $\rho_2 = \frac{\lambda_2}{\mu_2}$, $\delta_1 = \frac{\nu_1}{\mu_1}$, $\delta_2 = \frac{\nu_2}{\mu_2}$, $\gamma_1 = \frac{\mu_1^{/}}{\mu_1}$, and

$$\gamma_2 = \frac{\mu_2^{/}}{\mu_2} \tag{30}$$

It is clear that equations (27) and (28) are the solutions of single mode of failure. If S = 0, these equations represent the solution of single mode of failure without spares.

Case II : S < R

In this case the steady state equations are as follows:

$$-[M\lambda_1 + M\lambda_2]P(0,0) + \mu_1 P(1,0) + \mu_2 P(0,1) = 0 \tag{31}$$

$$-[M\lambda_1 + M\lambda_2 + i\mu_1]P(i, 0) + M\lambda_1 P(i-1, 0) + (i+1)\mu_1 P(i+1, 0) + \mu_2 P(i, 1) = 0,\ 1 \le i \le S \tag{32}$$

$$-[(N-i)\lambda_1 + M\lambda_2 + i\mu_1]P(i, 0) + (N-i+1)\lambda_1 P(i-1, 0) + (i+1)\mu_1 P(i+1, 0) + \mu_2 P(i, 1) = 0,\ S < i < R \tag{33}$$

$$-[(N-R)\beta_1\lambda_1 + M\lambda_2 + R\mu_1]P(R, 0) + (N-R+1)\lambda_1 P(R-1, 0) + (R\mu_1 + \nu_1)P(R+1, 0) + \mu_2 P(R,1) = 0 \tag{34}$$

$$-[(N-i)\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + (i-R)\nu_1]P(i, 0) + (N-i+1)\beta_1\lambda_1 P(i-1, 0) + [R\mu_1 + (i+1-R)\nu_1]P(i+1, 0) + \mu_2 P(i, 1) = 0,\ R < i < T \tag{35}$$

$$-[(N-T)\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + (T-R)\nu_1]P(T, 0) + (N-T+1)\beta_1\lambda_1 P(T-1, 0) + [R\mu_1 + \mu_1' + (T-R)\nu_1]P(T+1, 0) + \mu_2 P(T, 1) = 0, \tag{36}$$

$$-[(N-kT)\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + (k-1)\mu_1' + (kT-R-k+1)\nu_1]P(kT, 0) + (N-kT+1)\beta_1\lambda_1 P(kT-1, 0) + [R\mu_1 + k\mu_1' + (kT+1-R-k)\nu_1]P(kT+1, 0) + \mu_2 P(kT, 1) = 0,\ k = 1,2,...,r-1 \tag{37}$$

$$-[(N-rT)\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + (r-1)\mu_1' + (rT-R-r+1)\nu_1]P(rT, 0) + (N-rT+1)\beta_1\lambda_1 P(rT-1, 0) + [R\mu_1 + r\mu_1' + (rT+1-R-r)\nu_1]P(rT+1, 0) + \mu_2 P(rT, 1) = 0, \tag{38}$$

$-[(N-i)\beta_1\lambda_1 + M\lambda_2 + R\mu_1 + r\mu_1' + (i-R-r)\nu_1]P(i, 0) +$

$(N-i+1)\beta_1\lambda_1 P(i-1, 0)$
$+[R\mu_1+r\mu_1' + (i+1-R-r)\nu_1]P(i+1, 0) +$
$\mu_2 P(i, 1) = 0, \quad rT<i<N$ (39)

$-[R\mu_1+ r\mu_1' +(N-R-r)\nu_1]P(N, 0) + \beta_1\lambda_1 P(N-1, 0) = 0,$ (40)

$-[M\lambda_1+M\lambda_2 + j\mu_2]P(0, j) + M\lambda_2 P(0, j-1) + (j+1)\mu_2 P(0, j+1)$
$+ \mu_1 P(1, j) = 0, 1\le j \le S$ (41)

$-[(N-j)\lambda_2 +M\lambda_1 + j\mu_2]P(0, j) +(N-j+1)\lambda_2 P(0, j-1) +$
$(j+1)\mu_2 P(0, j+1)$
$+ \mu_1 P(1, j) = 0, S < j < R$ (42)

$-[(N-R)\beta_2\lambda_2 +M\lambda_1 + R\mu_2]P(0, R) + (N-R+1)\lambda_2 P(0, R-1) +$
$(R\mu_2+\nu_2)P(0, R+1) + \mu_1 P(1, R) = 0$ (43)

$-[(N-j) \beta_2\lambda_2 +M\lambda_1 + R\mu_2+(j-R)\nu_2]P(0, j) +$
$(N-j+1)\beta_2\lambda_2 P(0, j-1)$
$+[R\mu_2+ (j+1-R)\nu_2]P(0, j+1)+ \mu_1 P(1, j) = 0, R < i < T$ (44)

$-[M\lambda_1+(N-T) \beta_2\lambda_2 + R\mu_2+(T-R)\nu_2]P(0, T) +$
$(N-T+1)\beta_2\lambda_2 P(0, T-1)$
$+ [R\mu_2+\mu_2' + (T-R)\nu_2]P(0, T+1) + \mu_1 P(1, T) = 0,$ (45)

$-[M\lambda_1+(N-kT) \beta_2\lambda_2 + R\mu_2+(k-1)\mu_2' +(kT-R-k+1)\nu_2]P(0, kT)$
$+ (N-kT+1)\beta_2\lambda_2 P(0, kT-1)+$
$[R\mu_2+ k\mu_2' + (kT+1-R-k)\nu_2]P(0, kT+1)$
$+ \mu_1 P(1, kT) = 0, k = 1,2,...., r-1,$ (46)

$-[M\lambda_1+(N-rT) \beta_2\lambda_2 + R\mu_2+(r-1)\mu_2' +(rT-R-r+1)\nu_2]P(0, rT) +$
$(N-rT+1)\beta_2\lambda_2 P(0, rT-1) +$
$[R\mu_2+ r\mu_2' + (rT+1-R-r)\nu_2]P(0, rT+1) +$
$\mu_1 P(1, rT) = 0,$ (47)

$-[M\lambda_1+(N-j) \beta_2\lambda_2 + R\mu_2+r\mu_2' +(i-R-r)\nu_2]P(0, j) +$
$(N-j+1)\beta_2\lambda_2 P(0, j-1)$
$+ [R\mu_2+ r\mu_2' + (j+1-R-r)\nu_2]P(0, j+1) +$
$\mu_1 P(1, j) = 0, \quad rT< j < N$ (48)

$-[R\mu_2+ + r\mu_2' +(N-R-r)\nu_2]P(0, N) + \beta_2\lambda_2 P(0, N-1) = 0,$ (49)

$-[M\lambda_1+M\lambda_2+i\mu_1+j\mu_2]P(i, j) + M\lambda_1 P(i-1, j)+M\lambda_2 P(i, j-1)$
$+(i+1)\mu_1 P(i+1, j) + (j+1)\mu_2 P(i, j+1) = 0, i, j \ne 0, 1\le i+j\le S$ (50)

$-[(N-i-j)(\lambda_1+\lambda_2)+i\mu_1+j\mu_2]P(i, j) +$

$$(N-i-j+1)[\lambda_1 P(i-1,j)+\lambda_2 P(i,j-1)] +(i+1)\mu_1 P(i+1,j)+(j+1)\mu_2]P(i,j+1)=0,\ i,j\neq 0,\ S< i+j\leq R \tag{51}$$

$$-[(N-i-j)(\beta_1\lambda_1+\beta_2\lambda_2)+R\mu_1+(i-R)\nu_1+R\mu_2+(j-R)\nu_2]P(i,j)+[R\mu_1+(i+1-R)\nu_1]P(i+1,j)+[R\mu_2+(j+1-R)\nu_2]P(i,j+1)+(N-i-j+1)[\beta_1\lambda_1 P(i-1,j)+\beta_2\lambda_2 P(i,j-1)]=0,\ i,j\neq T,\ R<i+j\leq T \tag{52}$$

$$-[(N-i-j)(\beta_1\lambda_1+\beta_2\lambda_2)+R\mu_1+(k-1)\mu_1'+(i-R-k+1)\nu_1+R\mu_2+(k-1)\mu_2'+(j-R-k+1)\nu_2]P(i,j)+[R\mu_1+k\mu_1'+(i+1-R-k)\nu_1]P(i+1,j)+[R\mu_2+k\mu_2'+(j+1-R-k)\nu_2]P(i,j+1)+(N-i-j+1)[\beta_1\lambda_1 P(i-1,j)+\beta_2\lambda_2 P(i,j-1)]=0,\ i,j\neq kT,\ i+j=kT \tag{53}$$

$$-[(N-i-j)(\beta_1\lambda_1+\beta_2\lambda_2)+R\mu_1+k\mu_1'+(i-R-k)\nu_1+R\mu_2+k\mu_2'+(j-R-k)\nu_2]P(i,j)+[R\mu_1+(k+1)\mu_1'+(i-R-k)\nu_1]P(i+1,j)+[R\mu_2+(k+1)\mu_2'+(j-R-k)\nu_2]P(i,j+1)+(N-i-j+1)[\beta_1\lambda_1 P(i-1,j)+\beta_2\lambda_2 P(i,j-1)]=0,\ kT<i+j\leq(k+1)T \tag{54}$$

$$-[(N-i-j)(\beta_1\lambda_1+\beta_2\lambda_2)+R\mu_1+(r-1)\mu_1'+(i-R-r+1)\nu_1+R\mu_2+(r-1)\mu_2'+(j-R-r+1)\nu_2]P(i,j)+[R\mu_1+r\mu_1'+(i+1-R-r)\nu_1]P(i+1,j)+[R\mu_2+r\mu_2'+(j+1-R-r)\nu_2]P(i,j+1)+(N-i-j+1)[\beta_1\lambda_1 P(i-1,j)+\beta_2\lambda_2 P(i,j-1)]=0,\ i,j\neq rT,\ i+j=rT \tag{55}$$

$$-[(N-i-j)(\beta_1\lambda_1+\beta_2\lambda_2)+R\mu_1+r\mu_1'+(i-R-r)\nu_1+R\mu_2+r\mu_2'+(j-R-r)\nu_2]P(i,j)+[R\mu_1+r\mu_1'+(i+1-R-r)\nu_1]P(i+1,j)+[R\mu_2+r\mu_2'+(j+1-R-r)\nu_2]P(i,j+1)+(N-i-j+1)[\beta_1\lambda_1 P(i-1,j)+\beta_2\lambda_2 P(i,j-1)]=0,\ i,j\neq N,\ rT<i+j\leq N \tag{56}$$

The solution of equations (31)-(56) is in product form as

$$
P(i,0)=\begin{cases}
\dfrac{M^{i}\,\rho_1^{i}}{i\,!}P(0,0), & 1\le i\le S\\[2ex]
\dfrac{M^{S}\,M\,!\,\rho_1^{i}}{i\,!(N-i)!}P(0,0), & S<i\le R\\[2ex]
\dfrac{M^{S}\,M\,!\,\rho_1^{i}\,\beta_1^{i-R}}{R\,!\,(N-i)\,!\,\prod\limits_{n=R+1}^{i}[R+(n-R)\delta_1]}P(0,0), & R<i\le T\\[2ex]
\dfrac{M^{S}\,M\,!\,\rho_1^{i}\,\beta_1^{i-R}}{R\,!\,(N-i)\,!\,\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_1]\prod\limits_{l=1}^{k-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_1+(n-R-l)\delta_1]\prod\limits_{n=kT+1}^{i}[R+k\gamma_1+(n-R-k)\delta_1]}P(0,0), & kT<i\le(k+1)T,\ \ 1\le k<r\\[2ex]
\dfrac{M^{S}\,M\,!\,\rho_1^{i}\,\beta_1^{i-R}}{R\,!\,(N-i)\,!\,\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_1]\prod\limits_{l=1}^{r-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_1+(n-R-l)\delta_1]\prod\limits_{n=rT+1}^{i}[R+r\gamma_1+(n-R-r)\delta_1]}P(0,0), & rT<i\le N
\end{cases}
\tag{57}
$$

$$P(0,j)=\begin{cases} \dfrac{M^{j}\,\rho_2^{j}}{j\,!}P(0,0), & 1\le j\le S \\ \dfrac{M^{S}\,M!\,\rho_2^{j}}{j\,!(N-j)!}P(0,0), & S<j\le R \\ \dfrac{M^{S}\,M!\,\rho_2^{j}\,\beta_2^{j-R}}{R\,!\,(N-j)\,!\prod\limits_{n=R+1}^{j}[R+(n-R)\delta_2]}P(0,0), & R<j\le T \\ \dfrac{M^{S}\,M!\,\rho_2^{j}\,\beta_2^{j-R}}{R\,!\,(N-j)\,!\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_2]\prod\limits_{l=1}^{k-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_2+(n-R-l)\delta_2]\prod\limits_{n=kT+1}^{j}[R+k\gamma_2+(n-R-k)\delta_2]}P(0,0), & kT<j\le(k+1)T,\ \ 1\le k<r \\ \dfrac{M^{S}\,M!\,\rho_2^{j}\,\beta_2^{j-R}}{R\,!\,(N-j)\,!\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_2]\prod\limits_{l=1}^{r-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_2+(n-R-l)\delta_2]\prod\limits_{n=rT+1}^{j}[R+r\gamma_2+(n-R-r)\delta_2]}P(0,0), & rT<j\le N \end{cases} \tag{58}$$

$$P(i,j)=\begin{cases}\dfrac{M^{i+j}\rho_1^i\rho_2^j}{i!\,j!}P(0,0), & 1\le i+j\le S\\[2ex]\dfrac{M^S M!\rho_1^i\rho_2^j}{i!\,j!(N-i-j)!}P(0,0), & S<i+j\le R\\[2ex]\dfrac{M^S M!\,\rho_1^i\rho_2^j\beta_1^{i-R}\beta_2^{j-R}}{(R!)^2\,(N-i-j)!\prod\limits_{n=R+1}^{i}[R+(n-R)\delta_1]\prod\limits_{n=R+1}^{j}[R+(n-R)\delta_2]}P(0,0), & R<i+j\le T\\[2ex]\dfrac{M^S M!\,\rho_1^i\rho_2^j\beta_1^{i-R}\beta_2^{j-R}}{\begin{array}{c}(R!)^2(N-i-j)!\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_1]\prod\limits_{l=1}^{k-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_1+(n-R-l)\delta_1]\prod\limits_{n=kT+1}^{i}[R+k\gamma_1+(n-R-k)\delta_1]\\ \prod\limits_{n=R+1}^{T}[R+(n-R)\delta_2]\prod\limits_{l=1}^{k-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_2+(n-R-l)\delta_2]\prod\limits_{n=kT+1}^{j}[R+k\gamma_2+(n-R-k)\delta_2]\end{array}}P(0,0), & kT<i+j\le(k+1)T,\ 1\le k<r\\[2ex]\dfrac{M^S M!\,\rho_1^i\rho_2^j\beta_1^{i-R}\beta_2^{j-R}}{\begin{array}{c}(R!)^2(N-i-j)!\prod\limits_{n=R+1}^{T}[R+(n-R)\delta_1]\prod\limits_{l=1}^{r-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_1+(n-R-l)\delta_1]\prod\limits_{n=rT+1}^{i}[R+r\gamma_1+(n-R-r)\delta_1]\\ \prod\limits_{n=R+1}^{T}[R+(n-R)\delta_2]\prod\limits_{l=1}^{r-1}\prod\limits_{n=lT+1}^{(l+1)T}[R+l\gamma_2+(n-R-l)\delta_2]\prod\limits_{n=rT+1}^{j}[R+r\gamma_2+(n-R-r)\delta_2]\end{array}}P(0,0), & rT<i\le N\end{cases}\qquad(59)$$

We determine the value of P(0, 0) by using normalizing condition

$$\sum_{i=0}^{N}\sum_{j=0}^{N-i} P(i,j) = 1, \tag{60}$$

3. Some Performance Indices

The expected number of failed units of mode 1 in the system

$$E(N_1) = \sum_{i=1}^{N} i \sum_{j=0}^{N-i} P(i,j), \tag{61}$$

The expected number of failed units of mode 2 in the system

$$E(N_2) = \sum_{j=1}^{N} j \sum_{i=0}^{N-j} P(i,j), \tag{62}$$

The expected number of idle permanent repairmen

$$E(I) = \sum_{i=0}^{R-1} \sum_{j=0}^{R-1-i} P(i,j) = \sum_{i+j=0}^{R-1} (R-i-j)P(i,j) \tag{63}$$

The expected number of busy permanent repairmen in the system

$$E(B) = R - E(I) \tag{64}$$

The expected number of operating units in the system

$$E(O) = M - \sum_{i+j=S+1}^{N} (i+j-S)P(i,j), \tag{65}$$

The expected number of spare units in the system

$$E(S) = \sum_{i+j=0}^{S} (S-i-j)P(i,j) = \sum_{i=0}^{S}\sum_{j=0}^{S-i} (S-i-j)P(i,j), \tag{66}$$

The expected number of busy additional repairmen in the system

$$E(A)=\sum_{k=1}^{r-1} k \sum_{i+j=kT+1}^{(k+1)T} P(i,j)+r \sum_{i+j=rT+1}^{N} P(i,j), \tag{67}$$

Machine availability

$$\text{MA} = 1- \frac{E(N_1)+E(N_2)}{N} \tag{68}$$

Operative utilization

$$\text{OU} = \frac{E(B)}{R} = 1-\frac{E(I)}{R} \tag{69}$$

4. Cost Analysis

The aim of providing spare part support and additional repairmen is to maintain the system availability in the desired level. Our objective is to obtain the optimal number of repairmen and spares as R* and S* respectively. We assume that the costs per unit time are as follows:

- C_1 Cost per unit time of failed units when all spares are being used.
- C_2 Cost per unit time of a permanent repairman when it is idle.
- C_3 Cost per unit time of a permanent repairman when it is providing repair.
- C_4 Cost per unit time of providing a spare unit.
- C_5 Cost per unit time of an additional repairman.

The expected total cost per unit time is given by

$$E\{C(R, S)\} = C_1 \sum_{i+j=S+1}^{N}(i+j-S)P(i,j) + C_2 E(I) + C_3 E(B) + C_4 E(S) + C_5 E(A), \tag{70}$$

To obtain the optimal number of repairmen R* and the optimal number of spares S* in the system, the cost minimization problem (MP) is given by

$$\min_{R,S} \; Z = E\{C(R, S)\}, \tag{71}$$

subject to $$A_V = \sum_{i+j=0}^{S} P(i,j) = \sum_{i=0}^{S}\sum_{j=0}^{S-i} P(i,j) \geq A, \tag{72}$$

where A_V represents the probability that at least M units are operating under steady state and A is the minimum fraction of time when all M units are in operating mode to function properly. It is extremely difficult to determine the analytical solution for the optimal value of (R*, S*) as the expected total cost function is highly non-linear and complex. To solve the cost minimization problem (MP), a direct search method can be employed.

5. Conclusion

For the designing purpose, the main aim of the system designer is to investigate the model paying attention on system configuration so that reliability/ availability of the system can be achieved to a pre-assigned level. In this paper the steady state queue size distribution and other performance indices namely average number of failed units in the system, expected total cost per unit time are established for the machine interference model with balking, reneging and spares. The machine availability and the operative utilization are also determined. Various analytical results obtained are in explicit form, which may be computed easily. We have included the two types of failure so that model developed is closer to realistic situations. The provision of the spare part support as well as additional repairmen is done to improve the system efficiency. The balking and reneging factors used for system modeling make our model more feasible to the spectrum of production/ manufacturing systems in particular when failed units are discouraged due to backlog. The cost analysis facilitated may be helpful to system engineer in choosing the optimal combination of spares and repairmen. The developed model has many applications in the communication systems, manufacturing systems and production systems.

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تمثيل نظام انتظار مع الإعاقة والارتداد وعمال صيانة إضافيين وحالتي فشل

مادهي جان، جي سي شارما وماهفير سينغ

مدرسة علوم الرياضيات

معهد العلوم الأساسية – قانداري - أكرا - الهند

المستخلص . عادة ما يوصى بتوفير قطع الغيار لتحسين الاعتمادية في نظام الآلات . في هذا المقال ندرس مسألة إصلاح الآلة M/M/R مع الإعاقة والارتداد . ويفترض أن الوحدات تفشل في حالتين بنفس الاحتمال لكل حالة . كما يفترض عدم وجود وحدات (S) احتياطية لتحسين درجة الاعتمادية للسماحية . ويتم إصلاح الوحدة التي تفشل باستخدام تجهيزات تتكون من عدد (R) عمال صيانة دائمين وعدد آخر (V) من عمال الصيانة إضافيين، وإذا كان عدد الوحدات التي تفشل أكبر من عدد عمال الصيانة فلا يتم إصلاح الوحدة مباشرة . وفي هذه الحالة يمكن أن ترتد الوحدة أو تعاق. وتمثل مسألة إصلاح الآلات كنظام (نشأة – انتهاء) . ويتم الحصول على توزيع قائمة الانتظار في الحالة الثابتة بشكل مضروب، كما يتم الحصول على بعض مقاييس الأداء مثل العدد المتوقع للآلات التي تفشل والعدد المتوقع لعدد قطع الغيار ودرجة جودة الآلة . كما تم حساب التكاليف الإجمالية المتوقعة.

الكلمـات : قائمة الانتظار، مقاطعة الآلة، الإعاقة، الارتداد، قطع الغيار، الفشل، الإصلاح، الاعتمادية، تمثيل .